The structure of quantum Lie algebras for the classical series B_l, C_l and D_l

1998 J. Phys. A: Math. Gen. 31 1995

(http://iopscience.iop.org/0305-4470/31/8/012)

J. Phys. A: Math. Gen. **31** (1998) 1995–2019. Printed in the UK PII: S0305-4470(98)85449-8

The structure of quantum Lie algebras for the classical series B_l, C_l and D_l

Gustav W Delius†§, Christopher Gardner†‖ and Mark D Gould‡¶

† Department of Mathematics, King's College London, London WC2R 2LS, UK
‡ Department of Mathematics, University of Queensland, Brisbane QLD 4072, Australia

Received 30 June 1997

Abstract. The structure constants of quantum Lie algebras depend on a quantum deformation parameter q and they reduce to the classical structure constants of a Lie algebra at $q = 1$. We explain the relationship between the structure constants of quantum Lie algebras and quantum Clebsch–Gordan coefficients for adjoint ⊗ adjoint → adjoint. We present a practical method for the determination of these quantum Clebsch–Gordan coefficients and are thus able to give explicit expressions for the structure constants of the quantum Lie algebras associated to the classical Lie algebras B_l, C_l and D_l.

In the quantum case the structure constants of the Cartan subalgebra are non-zero and we observe that they are determined in terms of the simple quantum roots. We introduce an invariant Killing form on the quantum Lie algebras and find that it takes values which are simple q-deformations of the classical ones.

1. Introduction

Quantum Lie algebras are generalizations of Lie algebras whose structure constants depend on a quantum parameter q and which are related to the quantized enveloping algebras (quantum groups) $U_h(\mathfrak{g})$ in a way similar to how ordinary Lie algebras are related to their enveloping algebras $U(\mathfrak{g})$.

The study of quantum Lie algebras is still in its infancy. There is no fully developed theory of quantum Lie algebras yet. Instead, the properties of quantum Lie algebras are being discovered piecewise through detailed investigations of examples. It is hoped that these investigations will eventually lead to the development of a full theory. The process is similar to the development of Lie algebra theory which also began through the detailed study of the algebras of orthogonal, unitary and symplectic matrices.

In this paper we give the q-dependent structure constants of the quantum Lie algebras associated to the Lie algebras $so(n)$ of special orthogonal matrices and the Lie algebras $sp(n)$ of symplectic matrices. The case of $sl(n)$ had already been treated in [5].

Previous studies of quantum Lie algebras [4] have revealed that one could write the quantum Lie bracket relations in a form very similar to the classical ones. Namely, choosing a basis consisting of root generators X_α and Cartan subalgebra generators H_i the quantum

§ E-mail address: delius@mth.kcl.ac.uk. URL: http://www.mth.kcl.ac.uk/∼delius/
‖ E-mail address: cgardner@mth.kcl.ac.uk
¶ E-mail address: mdg@maths.uq.oz.au

Lie bracket relations are

$$
\begin{aligned}
&[H_i, X_\alpha]_h = l_\alpha(H_i) X_\alpha \qquad [X_\alpha, H_i]_h = -r_\alpha(H_i) X_\alpha \\
&[H_i, H_j]_h = \sum_k f_{ij}{}^k H_k \qquad [X_\alpha, X_{-\alpha}]_h = -\sum_k g_\alpha{}^k H_k \\
&[X_\alpha, X_\beta]_h = \begin{cases} N_{\alpha,\beta} X_{\alpha+\beta} & \text{if } \alpha+\beta \text{ is a root} \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
\tag{1.1}
$$

The main differences to the classical relations are:

- All structure constants now depend on the quantum parameter q.
- Every classical root α splits up into a left quantum root l_α and a right quantum root r_α which are related by $q \to 1/q$.
- The quantum Lie bracket $[H_i, H_j]_h$ between two Cartan subalgebra generators is in general non-zero.
- The quantum Lie bracket is q-antisymmetric.

As explained in the discussion section the results in this paper allow us to write these quantum Lie bracket relations using only the l_α and the $N_{\alpha\beta}$.

The notion of a quantum Lie algebra as an ad-submodule of the quantized enveloping algebra $U_h(\mathfrak{g})$ which transforms in the adjoint representation was introduced in [4]. In that paper it was shown that the structure constants of a quantum Lie algebra possess a number of symmetries, many of which are q-generalizations of their classical counterparts. These symmetries were derived by exploiting the Hopf algebra structure of $U_h(\mathfrak{g})$.

Another approach to quantum Lie algebras derives from the notion of bicovariant calculi on quantum groups [18, 1, 2]. There one defines a quantum Lie product on the dual space to the space of left-invariant one-forms, which is an ad-submodule of $U_h(\mathfrak{g})$. However, in this approach it was not known how to ensure that the resulting quantum Lie algebras have the same dimension as the corresponding classical Lie algebras except in the case of gl_n and, after projection, A_l [10, 15, 7].

In [5] the authors constructed the quantum Lie algebras associated to A_l. They first formed a submodule of $U_h(A_l)$ whose elements transform under the adjoint action (defined on $U_h(A_l)$) in the vector$\otimes$dual-vector representation. They then projected this module onto the adjoint representation inside vector$\otimes$dual-vector so that the resulting ad-module was of the correct dimension. While this approach can be applied to the construction of quantum Lie algebras associated to any simple Lie algebra $\mathfrak{g}$, it is a very tedious way of obtaining the structure constants and we present an alternative method here.

Our approach to the determination of the structure constants of quantum Lie algebras relies on the observation [6] that the quantum Lie bracket is an intertwiner from adjoint $\otimes$ adjoint $\to$ adjoint where by adjoint we mean the adjoint representation of $U_h(\mathfrak{g})$. Thus the structure constants are given by the corresponding inverse quantum Clebsch–Gordan coefficients. In this paper we describe a practical procedure for determining these Clebsch–Gordan coefficients. This allows us to explicitly calculate the structure constants of the quantum Lie algebras associated to the classical Lie algebras B_l, C_l and D_l.

This paper is organized as follows. In section 2 we recall the definitions and some properties of quantum Lie algebras. We then explain our method for determining their structure constants in section 3. In section 4 we introduce an invariant Killing form on the quantum Lie algebras and observe that on the basis vectors it takes values which are surprisingly simple deformations of the classical ones. Section 5 contains our results for the structure constants of the quantum Lie algebras. By using the Killing form to lower indices we are able to give concise expressions. In section 6 we derive some further relations between the structure constants, which we have verified using Mathematica [17] and which provide a good check on the correctness of our results. In the appendices we give the basic definition of the quantized enveloping algebras $U_h(\mathfrak{g})$ as algebras over $\mathbb{C}[[h]]$, the ring of formal power series in h. Further appendices contain a generalized form of Schur's lemma, some background material on Clebsch–Gordan coefficients and the quantum Lie algebra $(C_2)_h$.

2. Quantum Lie algebras

In this section we summarize the definitions for quantum Lie algebras given in [4, 5]. In these papers two definitions for quantum Lie algebras were given, the resulting algebras being isomorphic. The reason for the two definitions stems from the fact that we can view a classical Lie algebra $\mathfrak{g}$ in two different ways. Either we adopt the perspective that $\mathfrak{g}$ is the carrier space for the adjoint representation of the classical enveloping algebra $U(\mathfrak{g})$, this perspective leads to the so-called abstract quantum Lie algebras $\mathfrak{g}_h$. Alternatively, we view $\mathfrak{g}$ as a subspace of $U(\mathfrak{g})$. As will be seen later, the quantum Lie algebras $\mathfrak{L}_h(\mathfrak{g})$ deriving from the second definition are just embeddings of the abstract quantum Lie algebras $\mathfrak{g}_h$ into $U_h(\mathfrak{g})$.

2.1. Abstract quantum Lie algebras

Classically a Lie algebra $\mathfrak{g}$ is the carrier space for the adjoint representation $\pi^{\Psi}_{(0)}$ of the classical enveloping algebra $U(\mathfrak{g})$. The Lie bracket is a $U(\mathfrak{g})$-module homomorphism from $\mathfrak{g} \otimes \mathfrak{g}$ to $\mathfrak{g}$, i.e.

$$\pi^{\Psi}_{(0)}(x) \circ [\,,\,] = [\,,\,] \circ (\pi^{\Psi}_{(0)} \otimes \pi^{\Psi}_{(0)})(\Delta(x)) \qquad \forall x \in U(\mathfrak{g}). \tag{2.1}$$

Furthermore, the following theorem of Drinfel'd [8, 9] provides an isomorphism which allows the construction of the quantum analogue of the adjoint representation.

Theorem (Drinfel'd [8, 9]). There exists an algebra isomorphism $\varphi : U_h(\mathfrak{g}) \to U(\mathfrak{g})[[h]]$ such that $\varphi \equiv \mathrm{id}(\mathrm{mod}\, h)$ and $\varphi(h_i) = h_i$.

This implies that if (V^{μ}, π^{μ}) denotes a finite-dimensional irreducible representation of $U(\mathfrak{g})$ then $(V^{\mu}[[h]], \pi^{\mu} \circ \varphi)$ is a finite-dimensional indecomposable representation of $U_h(\mathfrak{g})$. (It is important to recognize that the $U_h(\mathfrak{g})$ modules $V[[h]]$ are not irreducible. Indeed their submodules are of the form $cV[[h]]$ with $c \in \mathbb{C}[[h]]$ not invertible and so Schur's lemma is modified, see appendix B.) In particular $\mathfrak{g}[[h]]$ is a finite-dimensional indecomposable module of $U_h(\mathfrak{g})$. Let $\pi^{\Psi} = \pi^{\Psi}_{(0)} \circ \varphi$ denote the representation of $U_h(\mathfrak{g})$ on $\mathfrak{g}[[h]]$. Drinfel'd has shown that this is the only way to deform the adjoint representation $\pi^{\Psi}_{(0)}$. With these observations in mind, a natural definition for quantum Lie algebras is as follows.

Definition 2.1. A *quantum Lie bracket* is a $U_h(\mathfrak{g})$-module homomorphism $[\,,\,]_h$: $\mathfrak{g}[[h]] \hat{\otimes} \mathfrak{g}[[h]] \to \mathfrak{g}[[h]]$ such that $[\,,\,]_h = [\,,\,](\mathrm{mod}\, h)$.

$\mathfrak{g}_h = (\mathfrak{g}[[h]], [\,,\,]_h)$ is a *quantum Lie algebra* (viewed as an algebra over $\mathbb{C}[[h]]$ with non-associative product $[\,,\,]_h$).

Such a $U_h(\mathfrak{g})$-module homomorphism $[\,,\,]_h$ is unique up to scalar multiples for $\mathfrak{g} \neq A_l$. The reason for this comes from the observation that classically the adjoint representation appears in the tensor product of two adjoint representations with unit multiplicity. Furthermore, the decomposition of a $U_h(\mathfrak{g})$ tensor product representation into indecomposable $U_h(\mathfrak{g})$ modules is described by the classical multiplicities of the decomposition of the corresponding $U(\mathfrak{g})$ tensor product representation into irreducible $U(\mathfrak{g})$ representations. Therefore, by the modified form of Schur's lemma, the homomorphism $[\ ,\ ]_h$ is unique (up to rescaling).

2.2. Quantum Lie algebras $\mathfrak{L}_h(\mathfrak{g})$ inside $U_h(\mathfrak{g})$

Alternatively $\mathfrak{g}$ can be viewed as a subspace of its enveloping algebra $U(\mathfrak{g})$ with the Lie bracket on this subspace given by the adjoint action of $U(\mathfrak{g})$. So another natural definition of a quantum Lie algebra is as an ad-submodule of $U_h(\mathfrak{g})$ with the quantum Lie bracket given by the adjoint action of $U_h(\mathfrak{g})$.

Definition 2.2. A *quantum Lie algebra* $\mathfrak{L}_h(\mathfrak{g})$ *inside* $U_h(\mathfrak{g})$ is a finite-dimensional indecomposable ad-submodule of $U_h(\mathfrak{g})$ endowed with a *quantum Lie bracket* $[a, b]_h = (\mathrm{ad}a)b$ such that:

(a) $\mathfrak{L}_h(\mathfrak{g})$ is a deformation of $\mathfrak{g}$, i.e. there is an algebra isomorphism $\mathfrak{g} \cong \mathfrak{L}_h(\mathfrak{g})|_{h=0}$

(b) $\mathfrak{L}_h(\mathfrak{g})$ is invariant under $\tilde{\theta}, \tilde{S}$ and any diagram automorphism τ, where $\sim$ denotes q-conjugation, θ the Cartan involution and S the antipode, see appendix A. A *weak quantum Lie algebra* $\mathfrak{l}_h(\mathfrak{g})$ is defined similarly but without the second condition.

By ad-submodule we mean that it is invariant under the adjoint action of $U_h(\mathfrak{g})$ which, in Sweedler's notation [16], is given by

$$(\mathrm{ad}x)y = \sum x_{(1)} y S(x_{(2)}) \qquad x, y \in U_h(\mathfrak{g}). \tag{2.2}$$

This adjoint action defines an infinite-dimensional representation of $U_h(\mathfrak{g})$ on itself. Notice that classically this definition for the adjoint action reduces to the usual commutator when restricted to the Lie algebra naturally embedded in $U(\mathfrak{g})$, giving rise to the classical adjoint representation $\pi^{\Psi}_{(0)}$.

Proposition 2.1. The adjoint action restricted to $\mathfrak{L}_h(\mathfrak{g}) \otimes \mathfrak{L}_h(\mathfrak{g})$ is a $U_h(\mathfrak{g})$-module homomorphism from $\mathfrak{L}_h(\mathfrak{g}) \otimes \mathfrak{L}_h(\mathfrak{g})$ to $\mathfrak{L}_h(\mathfrak{g})$.

Proof. We need to show that

$$\sum((\mathrm{ad}(\mathrm{ad}x_{(1)})a)((\mathrm{ad}x_{(2)})(b))) = (\mathrm{ad}x)((\mathrm{ad}a)(b)) \qquad \forall a, b \in \mathfrak{L}_h(\mathfrak{g})$$

which, using the fact that $[a, b]_h = (\mathrm{ad}a)b$, is equivalent to

$$(\mathrm{ad}x) \circ [\ ,\ ]_h = [\ ,\ ]_h \circ (\mathrm{ad} \otimes \mathrm{ad})(\Delta(x)) \qquad \forall x \in U_h(\mathfrak{g}).$$

The left-hand side of equation (2.2), using co-commutativity of the Hopf algebra, can be expressed as

$$\begin{aligned}\sum(\mathrm{ad}x_{(1)} a S(x_{(2)}))(x_{(3)} b S(x_{(4)})) &= \sum x_{(1)} a_{(1)} S(x_{(4)}) x_{(5)} b S(x_{(6)}) S(x_{(2)} a_{(2)} S(x_{(3)})) \\ &= \sum x_{(1)} a_{(1)} \epsilon(x_{(4)}) b S(x_{(5)}) S^2(x_{(3)}) S(a_{(2)}) S(x_{(2)}) \\ &= \sum x_{(1)} a_{(1)} b S(a_{(2)}) S(x_{(2)}) \\ &= (\mathrm{ad}x)((\mathrm{ad}a)b)\end{aligned}$$

where we have used the Hopf algebra property $S(y_{(1)}) y_{(2)} = \epsilon(y)$, and the fact that S is a Hopf algebra antiautomorphism. □

How are these quantum Lie algebras related to the abstract quantum Lie algebras defined in the previous section? This question is answered in the following proposition.

Proposition 2.2. All weak quantum Lie algebras $\mathfrak{l}_h(\mathfrak{g})$ inside $U_h(\mathfrak{g})$ are isomorphic to an abstract quantum Lie algebra $\mathfrak{g}_h$.

Proof. $\mathfrak{l}_h(\mathfrak{g})$ are finite-dimensional indecomposable $U_h(\mathfrak{g})$ modules. By condition (a) of definition 2.2, they carry a deformation of the representation of $U(\mathfrak{g})$ carried by $\mathfrak{g}$. As mentioned earlier, Drinfel'd showed that there is only one such deformation of the classical adjoint representation $\pi^{\Psi}_{(0)}$, namely the adjoint representation $\pi^{\Psi} = \pi^{\Psi}_{(0)} \circ \varphi$ carried by $\mathfrak{g}[[h]]$. Therefore $\mathfrak{l}_h(\mathfrak{g})$ is isomorphic to $\mathfrak{g}[[h]]$ as a $U_h(\mathfrak{g})$-module. Furthermore, by proposition 2.1 the product on $\mathfrak{l}_h(\mathfrak{g})$ is a $U_h(\mathfrak{g})$-module homomorphism. □

In other words, the weak quantum Lie algebras $\mathfrak{l}_h(\mathfrak{g})$ are those embeddings of $\mathfrak{g}[[h]]$ in $U_h(\mathfrak{g})$ on which the adjoint action of $U_h(\mathfrak{g})$ coincides with the adjoint representation π^{Ψ}.

2.3. *General properties of quantum Lie algebras*

Proposition 2.3. Given the grading of the classical Lie algebra by the set of non-zero roots R and zero, described by

$$\mathfrak{g} = \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \oplus \mathfrak{g}_0 \qquad [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta} \tag{2.3}$$

with $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} | \pi^{\Psi}_{(0)}(h_i)x = \alpha(h_i)x \forall h_i\}$, a quantum Lie algebra $\mathfrak{g}_h$ possesses the grading

$$\mathfrak{g}_h = \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha[[h]] \oplus \mathfrak{g}_o[[h]] \qquad [\mathfrak{g}_\alpha[[h]], \mathfrak{g}_\beta[[h]]]_h \subset \mathfrak{g}_{\alpha+\beta}[[h]]. \tag{2.4}$$

Proof. The isomorphism φ leaves h_i invariant. This implies $\pi^{\Psi}(h_i) = \pi^{\Psi}_{(0)}(\varphi(h_i)) = \pi^{\Psi}_{(0)}(h_i)$, so that

$$\mathfrak{g}_\alpha[[h]] = \{x \in \mathfrak{g}[[h]] | \pi^{\Psi}(h_i)x = \alpha(h_i)x \forall h_i\}. \tag{2.5}$$

Let $X_\alpha \in \mathfrak{g}_\alpha[[h]]$ and $X_\beta \in \mathfrak{g}_\beta[[h]]$. By definition, $[\ ,\ ]_h$ is a $U_h(\mathfrak{g})$-module homomorphism and so

$$\begin{aligned}\pi^{\Psi}(h_i)[X_\alpha, X_\beta]_h &= [\ ,\ ]_h \circ (\pi^{\Psi} \otimes \pi^{\Psi})(\Delta(h_i))(X_\alpha \hat{\otimes} X_\beta)\\ &= [\pi^{\Psi}(h_i)X_\alpha, X_\beta]_h + [X_\alpha, \pi^{\Psi}(h_i)X_\beta]_h\\ &= (\alpha(h_i) + \beta(h_i))[X_\alpha, X_\beta]_h\end{aligned} \tag{2.6}$$

thus $[X_\alpha, X_\beta]_h \in \mathfrak{g}_{\alpha+\beta}[[h]]$. □

If we now choose a basis for the quantum Lie algebra $\mathfrak{g}_h$, given by $\{X_\alpha \in \mathfrak{g}_\alpha | \alpha \in R\} \cup \{H_i \in \mathfrak{g}_0 | i = 1, \ldots \mathrm{rank}(\mathfrak{g})\}$, then, because of the grading, the Lie bracket is restricted to the following form:

$$\begin{aligned}&[H_i, X_\alpha]_h = l_\alpha(H_i)X_\alpha \qquad [X_\alpha, H_i]_h = -r_\alpha(H_i)X_\alpha\\ &[H_i, H_j]_h = \sum_k f^k_{ij} H_k \qquad [X_\alpha, X_{-\alpha}]_h = -\sum_k g^k_\alpha H_k\\ &[X_\alpha, X_\beta]_h = \begin{cases} N_{\alpha,\beta} X_{\alpha+\beta} & \text{for } \alpha + \beta \in R\\ 0 & \text{otherwise.}\end{cases}\end{aligned} \tag{2.7}$$

We call the zero-weight subalgebra $\mathcal{H} := \mathfrak{g}_0[[h]]$ of $\mathfrak{g}_h$ spanned by the generators $\{H_i\}_{i=1,\ldots,\mathrm{rank}\mathfrak{g}}$ the Cartan subalgebra even though, unlike in the classical case, $[H_i, H_j]_h \neq 0$.

The quantum roots l_α and r_α are linear functionals on $\mathcal{H}$. The structure constants $l_\alpha(H_i)$, $r_\alpha(H_i)$, g_α^k, ${f_{ij}}^k$ and $N_{\alpha\beta}$, are power series in the indeterminate h (in fact most of them turn out to be polynomials in $q^{1/2} = \mathrm{e}^{h/2}$).

The quantum structure constants possess a number of symmetries [4, 6], many of which are natural q-generalizations of their classical counterparts. The basis can be chosen so that the symmetries are given by:

$$l_\alpha = \tilde{r}_\alpha \qquad {f_{ij}}^k = -{\tilde{f}_{ji}}{}^k$$
$$N_{\alpha,\beta} = -\tilde{N}_{\beta,\alpha} \qquad g_\alpha^k = -\tilde{g}_{-\alpha}^k \tag{2.8}$$
$$l_\alpha = -\tilde{l}_{-\alpha} \qquad f_{ij}^k = -\tilde{f}_{ij}^k$$
$$N_{\alpha,\beta} = -\tilde{N}_{-\alpha,-\beta} \tag{2.9}$$
$$N_{\alpha,\beta} = q^{\rho\cdot\beta} N_{\beta,-\alpha-\beta} \tag{2.10}$$
$$\sum_l f_{jk}^l B(H_i, H_l) = \sum_l f_{ji}^l B(H_l, H_k) \tag{2.11}$$
$$-g_\alpha^i B(H_i, H_j) = l_\alpha(H_j) q^{-\rho\cdot\alpha} \tag{2.12}$$

where B is the Killing form, to be discussed in section 4, ρ is half the sum of the positive roots, and $\sim$ denotes q-conjugation, i.e. the operation $h \mapsto -h$ or equivalently $q \mapsto 1/q$. Note that in this paper we choose the basis for the quantum Lie algebra slightly differently from [4] to satisfy

$$B(X_\alpha, X_{-\alpha}) = q^{-\rho\cdot\alpha} \tag{2.13}$$
$$\tilde{\theta}(X_\alpha) = -X_{-\alpha} \qquad \tilde{\theta}(H_i) = -H_i \tag{2.14}$$
$$\tilde{S}(X_\alpha) = -q^{\rho\cdot\alpha} X_\alpha \qquad \tilde{S}(H_i) = -H_i. \tag{2.15}$$

The symmetries in (2.8) express the q-antisymmetry of the quantum Lie bracket

$$[a, b]_h^\triangledown = -[b^\triangledown, a^\triangledown]_h. \tag{2.16}$$

The map $^\triangledown : \mathfrak{g}[[h]] \to \mathfrak{g}[[h]]$ is defined so that if $a = \sum_\alpha \zeta_\alpha X_\alpha + \sum_i \eta_i H_i$ is a general element of $\mathfrak{g}[[h]]$, then $a^\triangledown = \sum_\alpha \tilde{\zeta}_\alpha X_\alpha + \sum_i \tilde{\eta}_i H_i$. In the classical limit q-antisymmetry expresses the antisymmetry of the Lie bracket. The origin of the q-antisymmetry of the quantum Lie bracket was explained in [6].

2.4. *Quantum structure constants as quantum Clebsch–Gordan coefficients*

Let $\{v^a\}_{a=1,\ldots\dim\mathfrak{g}}$ be a basis for $\mathfrak{g}[[h]]$. The highest weight vector v^1 generating this $U_h(\mathfrak{g})$-module satisfies the relations,

$$\pi^\Psi(x_i^+)v^1 = 0 \qquad \pi^\Psi(h_i)v^1 = \Psi(h_i)v^1 \qquad \forall i \tag{2.17}$$

where Ψ is the highest root of $\mathfrak{g}$ and $\pi^\Psi = \pi^\Psi_{(0)} \circ \varphi$. Let $P^a(x^-)$ be polynomials in the x_i^- such that $v^a = \pi^\Psi(P^a(x^-))v^1$. The adjoint representation matrices in this basis are given by

$$\pi^\Psi(x)v^a = v^b \pi_b^{\Psi a}(x). \tag{2.18}$$

Notation. Latin indices which appear as both upper and lower indices in the same expression are summed over.

We know that $\mathfrak{g}[[h]]\hat{\otimes}\mathfrak{g}[[h]]$ contains $\mathfrak{g}[[h]]$ with unit multiplicity for the algebras B_l, C_l and D_l. Hence, there exists a highest weight state $\hat{v}^1$ inside $\mathfrak{g}[[h]]\hat{\otimes}\mathfrak{g}[[h]]$ that satisfies the analogue of relations (2.17), namely

$$(\pi^\Psi \otimes \pi^\Psi)(\Delta(x_i^+))\hat{v}^1 = 0 \qquad (\pi^\Psi \otimes \pi^\Psi)(\Delta(h_i))\hat{v}^1 = \Psi(h_i)\hat{v}^1 \qquad \forall i \tag{2.19}$$

and $\hat{v}^1$ is the unique (up to rescaling) state with this property. The vectors $\hat{v}^a = (\pi^\Psi \otimes \pi^\Psi)(\Delta(P^a(x^-)))\hat{v}^1$ form a basis for $\mathfrak{g}[[h]]$ inside $\mathfrak{g}[[h]]\hat{\otimes}\mathfrak{g}[[h]]$ such that

$$(\pi^\Psi \otimes \pi^\Psi)(\Delta(x))\hat{v}^a = \hat{v}^b \pi^{\Psi a}_b(x) \tag{2.20}$$

with the same representation matrices as in (2.18). We can expand the vectors $\hat{v}^a$ in terms of the tensor product basis as

$$\hat{v}^a = v^b \otimes v^c C^{\Psi\Psi}_{b\ c}|^a_\Psi. \tag{2.21}$$

The $C^{\Psi\Psi}_{b\ c}|^a_\Psi \in \mathbb{C}[[h]]$ are the Clebsch–Gordan coefficients. In the notation we often indicate above or below an index which representation the index belongs to. We do this by giving the highest weight of the representation, e.g. in $C^{\Psi\Psi}_{b\ c}|^a_\Psi$ all indices belong to the representation with highest weight Ψ (Ψ being the highest root of the Lie algebra), i.e. the adjoint representation.

The embedding map $\beta : \mathfrak{g}[[h]] \to \mathfrak{g}[[h]]\hat{\otimes}\mathfrak{g}[[h]]$, defined by $v^a \to \hat{v}^a$ is a $U_h(\mathfrak{g})$-module homomorphism, i.e. $\beta \circ \pi^\Psi(x) = (\pi^\Psi \otimes \pi^\Psi)(\Delta(x)) \circ \beta$. This is easy to check:

$$\begin{aligned}(\beta \circ \pi^\Psi(x))v^a &= \beta(v^b\pi^{\Psi a}_b(x)) = \hat{v}^b\pi^{\Psi a}_b(x) = (\pi^\Psi \otimes \pi^\Psi)(\Delta(x))\hat{v}^a \\ &= ((\pi^\Psi \otimes \pi^\Psi)(\Delta(x)) \circ \beta)v^a.\end{aligned}$$

Both $\mathfrak{g}[[h]]$ and $\mathrm{Im}\,\beta$ are indecomposable modules and so β (with its range restricted to $\mathrm{Im}\,\beta$) is unique and invertible by the weak form of Schur's lemma. We now define the quantum Lie bracket $[\ ,\ ]_h : \mathfrak{g}[[h]]\hat{\otimes}\mathfrak{g}[[h]] \to \mathfrak{g}[[h]]$ to be zero on the module complement of $\mathrm{Im}\,\beta$ whilst on $\mathrm{Im}\,\beta$ we define $[\ ,\ ]_h = \beta^{-1}$. Then

$$[v^a, v^b] = C^{\ \Psi}_c|^{a\ b}_{\Psi\Psi} v^c \qquad \text{with } C^{\ \Psi}_d|^{b\ c}_{\Psi\Psi} C^{\Psi\Psi}_{b\ c}|^a_\Psi = \delta^a_d. \tag{2.22}$$

Finally, because β is a $U_h(\mathfrak{g})$-module homomorphism, β^{-1} is also a $U_h(\mathfrak{g})$-module homomorphism. So we have shown that we can construct a bracket with the correct properties.

Conclusion. To construct the quantum Lie algebra $\mathfrak{g}_h$ associated to $\mathfrak{g}$ we need to calculate the inverse Clebsch–Gordan coefficients for the decomposition $\mathfrak{g}[[h]]\hat{\otimes}\mathfrak{g}[[h]]$ into $\mathfrak{g}[[h]]$.

3. Calculation of quantum Clebsch–Gordan coefficients

In this section we present our method for the calculation of the inverse quantum Clebsch–Gordan coefficients for adjoint $\otimes$ adjoint $\to$ adjoint. We do not actually form adjoint $\otimes$ adjoint. Instead we build the Clebsch–Gordan coefficients from vector $\otimes$ vector. The reason for this lies in the relative simplicity of the vector representations of $U_h(\mathfrak{g})$ for B_l, C_l and D_l. The most attractive feature of the vector representation is that the classical representation matrices for the Cartan subalgebra generators have eigenvalues $+1, 0$ or -1. This means that the classical representation matrices are also representation matrices for $U_h(\mathfrak{g})$.

Let us first remind ourselves of the roots for the algebras B_l, C_l and D_l. Let ϵ_i denote the orthonormal basis vectors of the root space Φ. The roots for B_l can be written as $\pm(\epsilon_i \pm \epsilon_j)$ with $i \neq j$ and $\pm\epsilon_i$ with $i \leqslant l$. The simple roots are $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $i < l$

and $\alpha_l = \epsilon_l$. For C_l the roots are given by $2\epsilon_i$ with $i \leqslant l$ and $\pm(\epsilon_i \pm \epsilon_j)$ with $i \neq j$. The simple roots are $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $i < l$ and $\alpha_l = 2\epsilon_l$. For D_l the roots are given by $\pm(\epsilon_i \pm \epsilon_j)$ with $i \neq j$ and the simple roots are $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $i < l$ and $\alpha_l = \epsilon_{l-1} + \epsilon_l$. The vector representation is generated from the highest weight state, denoted by $|1\rangle$, whose weight is ϵ_1. This corresponds to $\alpha_1 + \alpha_2 + \cdots + \alpha_l$ for B_l, $\alpha_1 + \alpha_2 + \cdots + \frac{1}{2}\alpha_l$ for C_l, and $\alpha_1 + \alpha_2 + \cdots + \frac{1}{2}(\alpha_{l-1} + \alpha_l)$ for D_l. The value of $\rho \cdot \alpha$ is calculated easily from the value of $\rho \cdot \epsilon_i$ $(i \leqslant l)$ as given below.

$$\rho \cdot \epsilon_i = \begin{cases} l - i + \frac{1}{2} & \text{for } B_l \\ l - i + 1 & \text{for } C_l \\ l - i & \text{for } D_l. \end{cases} \tag{3.1}$$

In the following we will denote by V^μ the representation space with highest weight μ, e.g. V^Ψ is the adjoint representation, V^{ϵ_1} the vector representation and V^0 the singlet representation.

3.1. The Clebsch–Gordan coefficients for adjoint $\otimes$ adjoint $\rightarrow$ adjoint

To calculate these Clebsch–Gordan coefficients we start from the simpler decomposition of the direct product representation $V^{\epsilon_1} \otimes V^{\epsilon_1} \rightarrow V^{2\epsilon_1} \oplus V^{\epsilon_1+\epsilon_2} \oplus V^0$. We denote by $\{v^a_\mu\}_{a=1,\dots,\dim V^\mu}$ the basis of a representation space V^μ. Then the decomposition is described by the inverse Clebsch–Gordan coefficients as follows

$$v^a_{\epsilon_1} \otimes v^b_{\epsilon_1} = \sum_{c=1}^{\dim V^{2\epsilon_1}} v^c_{2\epsilon_1} C^{2\epsilon_1}_c|^{a\ b}_{\epsilon_1 \epsilon_1} + \sum_{c=1}^{\dim V^{\epsilon_1+\epsilon_2}} v^c_{\epsilon_1+\epsilon_2} C^{\epsilon_1+\epsilon_2}_c|^{a\ b}_{\epsilon_1 \epsilon_1} + v_0 C^0|^{a\ b}_{\epsilon_1 \epsilon_1}. \tag{3.2}$$

It is very convenient to introduce a graphical notation for the Clebsch–Gordan coefficients and their inverses as in figure 1.

The composition of any number of intertwiners is again an intertwiner. We exploit this fact to build the intertwiner for adjoint $\otimes$ adjoint $\rightarrow$ adjoint from the intertwiners for

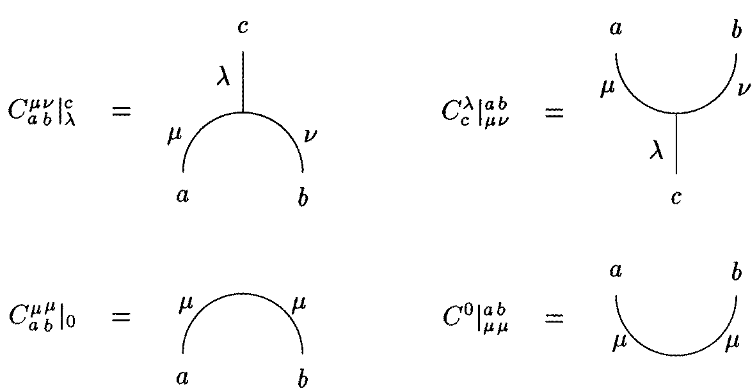
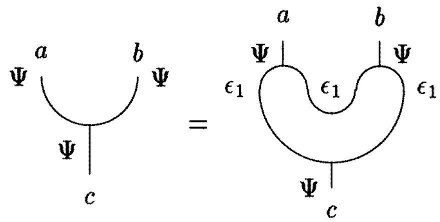


Figure 1. Graphical representation of Clebsch–Gordan coefficients.

vector ⊗ vector into adjoint and singlet as shown in the following diagram



(3.3)

or equivalently

$$C^{\Psi}_{c}|^{a\ b}_{\Psi\,\Psi} = C^{\epsilon_1\epsilon_1}_{i\ \ j}|^{a}_{\Psi} C^{\epsilon_1\epsilon_1}_{k\ \ l}|^{b}_{\Psi} C^{0}|^{j\ k}_{\epsilon_1\epsilon_1} C^{\Psi}_{c}|^{i\ \ l}_{\epsilon_1\epsilon_1}. \quad (3.4)$$

Let us show explicitly that the $C^{\Psi}|_{\Psi\Psi}$ defined by this equation does indeed satisfy the intertwining property

$$C^{\Psi}_{c}|^{a'b'}_{\Psi\,\Psi} \pi^{\Psi a}_{a'}(x_{(1)}) \pi^{\Psi b}_{b'}(x_{(2)}) = \pi^{\Psi c'}_{c}(x) C^{\Psi}_{c'}|^{a\ b}_{\Psi\,\Psi} \qquad \forall x \in U_h(\mathfrak{g}). \quad (3.5)$$

Substituting (3.4) and using the intertwining property of $C^{\epsilon_1\epsilon_1}|_{\Psi}$, the left-hand side of (3.5) becomes

$$\pi^{\epsilon_1 i'}_{i}(x_{(1)}) \pi^{\epsilon_1 j'}_{j}(x_{(2)}) C^{\epsilon_1\epsilon_1}_{i'\ j'}|^{a}_{\Psi} \pi^{\epsilon_1 k'}_{k}(x_{(3)}) \pi^{\epsilon_1 l'}_{l}(x_{(4)}) C^{\epsilon_1\epsilon_1}_{k'\ l'}|^{b}_{\Psi} C^{0}|^{j\ k}_{\epsilon_1\epsilon_1} C^{\Psi}_{c}|^{i\ \ l}_{\epsilon_1\epsilon_1}. \quad (3.6)$$

The one-dimensional representation of $U_h(\mathfrak{g})$ is given by the co-unit ϵ (recall $\epsilon(xy) = \epsilon(x)\epsilon(y) \forall x, y \in U_h(\mathfrak{g})$), and so

$$C^{0}|^{j\ k}_{\epsilon_1\epsilon_1} \pi^{\epsilon_1 j'}_{j}(x_{(2)}) \pi^{\epsilon_1 k'}_{k}(x_{(3)}) = \epsilon(x_{(2)}) C^{0}|^{j'k'}_{\epsilon_1\epsilon_1}. \quad (3.7)$$

Consequently one obtains (using co-commutativity)

$$C^{\epsilon_1\epsilon_1}_{i'\ j'}|^{a}_{\Psi} C^{\epsilon_1\epsilon_1}_{k'\ l'}|^{b}_{\Psi} C^{0}|^{j'k'}_{\epsilon_1\epsilon_1} C^{\Psi}_{c}|^{i\ \ l}_{\epsilon_1\epsilon_1} \pi^{\epsilon_1 i'}_{i}(x_{(1)}) \epsilon(x_{(2)}) \pi^{\epsilon_1 l'}_{l}(x_{(3)}) \quad (3.8)$$

which is equivalent to the right-hand side of (3.5), if we use the intertwining property of the Clebsch–Gordan coefficient $C^{\Psi}|_{\epsilon_1\epsilon_1}$ and the co-unit property $\epsilon(x_{(1)})x_{(2)} = x$.

3.2. The vector representation V^{ϵ_1}

We now turn our attention to the vector representation. We change the notation and denote the basis states for the vector representation by $|i\rangle$ with $i = 1, \ldots, 2l$ for C_l and D_l, and $i = 1, \ldots, 2l+1$ for B_l. The matrices e_{ij} act on these basis states as $e_{ij}|k\rangle = \delta_{jk}|i\rangle$.

The states in the vector representation have the following weights:

$$\begin{aligned} &|i\rangle \longleftrightarrow \epsilon_i \qquad i \leqslant l \\ &|l+1\rangle \longleftrightarrow 0 \\ &|i\rangle \longleftrightarrow -\epsilon_{\bar{i}} \qquad l+1 < i \leqslant 2l+1 \end{aligned}$$

for the algebra B_l ($\bar{i} = 2l+2-i$), and

$$\begin{aligned} &|i\rangle \longleftrightarrow \epsilon_i \qquad i \leqslant l \\ &|i\rangle \longleftrightarrow -\epsilon_{\bar{i}} \qquad l < i \leqslant 2l \end{aligned}$$

for the algebras C_l and D_l ($\bar{i} = 2l+1-i$).

In the following the E_i, F_i and H_i are abbreviations for $\pi^{\epsilon_1}(x_i^+)$, $\pi^{\epsilon_1}(x_i^-)$ and $\pi^{\epsilon_1}(h_i)$.

3.2.1. $\mathfrak{g} = B_l$.

$$E_i = a_{i,i+1} \qquad F_i = a_{i+1,i} \qquad H_i = a_{ii} - a_{i+1,i+1} \qquad 1 \leqslant i < l$$
$$E_l = \sqrt{2} a_{l,l+1} \qquad F_l = \sqrt{2} a_{l+1,l} \qquad H_l = 2a_{l,l} \tag{3.9}$$

where

$$a_{ij} = e_{ij} - e_{\overline{ji}} \qquad \bar{i} = 2l + 2 - i. \tag{3.10}$$

3.2.2. $\mathfrak{g} = C_l$.

$$E_i = a_{i,i+1} \qquad F_i = a_{i+1,i} \qquad H_i = a_{ii} - a_{i+1,i+1} \qquad 1 \leqslant i < l$$
$$E_l = \tfrac{1}{2} a_{l,l+1} \qquad F_l = \tfrac{1}{2} a_{l+1,l} \qquad H_l = a_{l,l} \tag{3.11}$$

where

$$a_{ij} = e_{ij} - (-1)^{i+j} e_{\overline{ji}} \qquad \bar{i} = 2l + 1 - i.$$

3.2.3. $\mathfrak{g} = D_l$.

$$E_i = a_{i,i+1} \qquad F_i = a_{i+1,i} \qquad H_i = a_{ii} - a_{i+1,i+1} \qquad 1 \leqslant i < l$$
$$E_l = a_{l-1,l+1} \qquad F_l = a_{l+1,l-1} \qquad H_l = a_{l-1,l-1} - a_{l+1,l+1} \tag{3.12}$$

where

$$a_{ij} = e_{ij} - e_{\overline{ji}} \qquad \bar{i} = 2l + 1 - i.$$

3.3. Direct product representation $V^{\epsilon_1} \otimes V^{\epsilon_1}$

The direct product representation is constructed in the usual manner via the coproduct. For all the algebras B_l, C_l and D_l, $V^{\epsilon_1} \otimes V^{\epsilon_1}$ has the following decomposition:

$$V^{\epsilon_1} \otimes V^{\epsilon_1} = V^{2\epsilon_1} \oplus V^{\epsilon_1+\epsilon_2} \oplus V^0 \tag{3.13}$$

where $V^{\epsilon_1+\epsilon_2}$ is the adjoint representation for B_l and D_l, and $V^{2\epsilon_1}$ is the adjoint representation for C_l. In the following we give basis vectors for these submodules. The expansion coefficients of these basis vectors in terms of the V^{ϵ_1} basis vectors give the quantum Clebsch–Gordan coefficients (cf (3.2)). We have chosen the basis of the vector representation to be self-dual, namely $\langle i | j \rangle = \delta_{ij}$. Therefore the expansion coefficients of the dual basis are the inverse quantum Clebsch–Gordan coefficients. We will use a generalized Kronecker delta notation; for example, $\delta_{i<j} = 1$ if $i < j$, 0 otherwise.

3.3.1. $\mathfrak{g} = B_l$.

- Basis for $V^{2\epsilon_1}$:

$$\omega_{ij} = q^{1/2} |i\rangle \otimes |j\rangle + q^{-1/2} |j\rangle \otimes |i\rangle \qquad i < j \neq \bar{i}$$
$$\omega_{ii} = |i\rangle \otimes |i\rangle \qquad 1 \leqslant i \neq l+1 \leqslant 2l$$
$$\omega_i = q|i\rangle \otimes |\bar{i}\rangle + q^{-1} |\bar{i}\rangle \otimes |i\rangle - (|i+1\rangle \otimes |\overline{i+1}\rangle + |\overline{i+1}\rangle \otimes |i+1\rangle)\delta_{i<l}$$
$$- (q^{1/2} + q^{-1/2}) |l+1\rangle \otimes |l+1\rangle \delta_{il} \qquad i \leqslant l$$

with corresponding dual basis

$$\omega^i = \frac{(q+q^{-1})^{-1}}{D_l} \Bigg(D_{l-i} \sum_{j \leqslant i} (q^j \langle j| \otimes \langle \bar{j}| + q^{-j} \langle \bar{j}| \otimes \langle j|) - [i]_q (q^{1/2} + q^{-1/2})$$
$$\times \Bigg(\langle l+1| \otimes \langle l+1| - \sum_{j>i}^{l} (q^{j-l-1/2} \langle j| \otimes \langle \bar{j}| + q^{l+1/2-j} \langle \bar{j}| \otimes \langle j|) \Bigg) \Bigg)$$

where

$$D_k = \frac{q^{k+1/2} - q^{-k-1/2}}{q^{1/2} - q^{-1/2}}$$

and the q-numbers are given by

$$[i]_q = \frac{q^i - q^{-i}}{q - q^{-1}}.$$

- Basis for $V^{\epsilon_1+\epsilon_2}$:

$$v_{ij} = q^{-1/2}|i\rangle \otimes |j\rangle - q^{+1/2}|j\rangle \otimes |i\rangle \qquad i < j \neq \bar{i}$$

$$v_i = |i\rangle \otimes |\bar{i}\rangle - |\bar{i}\rangle \otimes |i\rangle - (q^{-1}|i+1\rangle \otimes |\overline{i+1}\rangle - q|\overline{i+1}\rangle \otimes |i+1\rangle)\delta_{i<l} + (q^{1/2} - q^{-1/2})|l+1\rangle \otimes |l+1\rangle\delta_{il} \qquad i \leqslant l$$

with corresponding dual basis

$$v^i = \frac{(q+q^{-1})^{-1}}{D_l}\Bigg([i]_q(q^{1/2} - q^{-1/2})\Bigg(\langle l+1| \otimes \langle l+1| + \sum_{j>i}^{l}(q^{j-l-1/2}\langle j| \otimes \langle\bar{j}| + q^{l+1/2-j}\langle\bar{j}| \otimes \langle j|)\Bigg) + D_{l-i}\sum_{j\leqslant i}(q^{j-1}\langle j| \otimes \langle\bar{j}| - q^{1-j}\langle\bar{j}| \otimes \langle j|)\Bigg)$$

where

$$D_k = \frac{q^{k-1/2} + q^{1/2-k}}{q^{1/2} + q^{-1/2}}.$$

- Basis for V^0:

$$t = \sum_{i=1}^{l}(q^{i-l-1/2}|i\rangle \otimes |\bar{i}\rangle + q^{-i+l+1/2}|\bar{i}\rangle \otimes |i\rangle) + |l+1\rangle \otimes |l+1\rangle.$$

3.3.2. $\mathfrak{g} = C_l$.

- Basis for $V^{2\epsilon_1}$:

$$v_{ij} = q^{1/2}|i\rangle \otimes |j\rangle + q^{-1/2}|j\rangle \otimes |i\rangle \qquad i < j \neq \bar{i}$$

$$v_{ii} = |i\rangle \otimes |i\rangle \qquad 1 \leqslant i \leqslant 2l$$

$$v_i = (-1)^i(q|i\rangle \otimes |\bar{i}\rangle + q^{-1}|\bar{i}\rangle \otimes |i\rangle + (|i+1\rangle \otimes |\overline{i+1}\rangle + |\overline{i+1}\rangle \otimes |i+1\rangle)\delta_{i<l}) \qquad 1 \leqslant i \leqslant l$$

whose duals are given by

$$v^i = \frac{(q+q^{-1})^{-1}}{D_l}\Bigg(D_{l-i}\sum_{j\leqslant i}(-1)^j(q^j\langle j| \otimes \langle\bar{j}| + q^{-j}\langle\bar{j}| \otimes \langle j|) + [i]_q\frac{q-q^{-1}}{q+q^{-1}}\sum_{j>i}^{l}(-1)^j(q^{j-l-1}\langle j| \otimes \langle\bar{j}| - q^{l+1-j}\langle\bar{j}| \otimes \langle j|)\Bigg)$$

where

$$D_k = \frac{q^{k+1} + q^{-k-1}}{q+q^{-1}}.$$

Basis for $V^{\epsilon_1+\epsilon_2}$:

$$\omega_{ij} = q^{-1/2}|i\rangle \otimes |j\rangle - q^{1/2}|j\rangle \otimes |i\rangle \qquad i < j \neq \bar{i}$$

$$\omega_i = (-1)^i(|i\rangle \otimes |\bar{i}\rangle - |\bar{i}\rangle \otimes |i\rangle + q^{-1}|i+1\rangle \otimes |\overline{i+1}\rangle - q|\overline{i+1}\rangle \otimes |i+1\rangle) \qquad 1 \leqslant i < l$$

whose corresponding duals are given by

$$\omega^i = \frac{(q+q^{-1})^{-1}}{[l]_q}\left([l-i]_q \sum_{j\leqslant i}(-1)^j(q^{j-1}\langle j| \otimes \langle\bar{j}| - q^{1-j}\langle\bar{j}| \otimes \langle j|) + [i]_q \sum_{j>i}^{l}(-1)^{j-1}(q^{j-l-1}\langle j| \otimes \langle\bar{j}| - q^{l+1-j}\langle\bar{j}| \otimes \langle j|)\right).$$

- Basis for V^0:

$$t = \sum_{i=1}^{l}(-1)^{l-i}(q^{i-l-1}|i\rangle \otimes |\bar{i}\rangle - q^{l+1-i}|\bar{i}\rangle \otimes |i\rangle).$$

3.3.3. $\mathfrak{g} = D_l$.

- Basis for $V^{2\epsilon_1}$:

$$\omega_{ii} = |i\rangle \otimes |i\rangle \qquad 1 \leqslant i \leqslant 2l$$

$$\omega_{ij} = q^{1/2}|i\rangle \otimes |j\rangle + q^{-1/2}|j\rangle \otimes |i\rangle \qquad i < j \neq \bar{i}$$

$$\omega_i = q|i\rangle \otimes |\bar{i}\rangle + q^{-1}|\bar{i}\rangle \otimes |i\rangle - (|i+1\rangle \otimes |\overline{i+1}\rangle + |\overline{i+1}\rangle \otimes |i+1\rangle) \qquad 1 \leqslant i < l$$

with corresponding duals

$$\omega^i = \frac{(q+q^{-1})^{-1}}{[l]_q}\left([l-i]_q \sum_{j\leqslant i}(q^{j}\langle j| \otimes \langle\bar{j}| + q^{-j}\langle\bar{j}| \otimes \langle j|) - [i]_q \sum_{j>i}^{l}(q^{j-l}\langle j| \otimes \langle\bar{j}| + q^{l-j}\langle\bar{j}| \otimes \langle j|)\right).$$

Basis for $V^{\epsilon_1+\epsilon_2}$:

$$v_{ij} = q^{-1/2}|i\rangle \otimes |j\rangle - q^{1/2}|j\rangle \otimes |i\rangle \qquad i < j \neq \bar{i}$$

$$v_i = |i\rangle \otimes |\bar{i}\rangle - |\bar{i}\rangle \otimes |i\rangle - (q^{-1}|i+1\rangle \otimes |\overline{i+1}\rangle - q|\overline{i+1}\rangle \otimes |i+1\rangle)\delta_{i<l},$$

with corresponding duals

$$v^i = \frac{(q+q^{-1})^{-1}}{D_l}\left(D_{l-i} \sum_{j\leqslant i}(q^{j-1}\langle j| \otimes \langle\bar{j}| - q^{1-j}\langle\bar{j}| \otimes \langle j|) + \frac{[i]_q(q-q^{-1})}{(q+q^{-1})} \sum_{j>i}^{l}(q^{j-l}\langle j| \otimes \langle\bar{j}| + q^{l-j}\langle\bar{j}| \otimes \langle j|)\right)$$

where

$$D_k = \frac{(q^{k-1}+q^{1-k})}{(q+q^{-1})}.$$

- Basis for V^0:

$$t = \sum_{i=1}^{l}(q^{i-l}|i\rangle \otimes |\bar{i}\rangle + q^{l-i}|\bar{i}\rangle \otimes |i\rangle).$$

Nearly all the elements are in place to construct the quantum structure constants. It only remains to identify which vectors in the adjoint representation (embedded in $V^{\epsilon_1} \otimes V^{\epsilon_1}$) correspond to the Cartan subalgebra and which to the roots. We have already discussed the weights of the basis vectors of the vector representation and therefore we can calculate trivially the weights of the basis vectors of the adjoint representation. This then fixes the identification between the basis states of the adjoint representation given above and the basis states for the quantum Lie algebra chosen in (1.1) up to rescaling. We have chosen the scalings with hindsight so that the Killing form and the structure constants are as simple as possible. We set $\xi = (q + q^{-1})\langle t|t\rangle$.

For B_l we have

$$\begin{aligned} &X_{\epsilon_i-\epsilon_j} = \xi\, v_{i\bar{j}} \qquad X_{-(\epsilon_i-\epsilon_j)} = \xi\, v_{j\bar{i}} \\ &X_{\epsilon_i+\epsilon_j} = \xi\, v_{ij} \qquad X_{-(\epsilon_i+\epsilon_j)} = \xi\, v_{\overline{ji}} \\ &X_{\epsilon_j} = \xi\, v_{jl+1} \qquad X_{-\epsilon_j} = \xi\, v_{l+1\bar{j}} \\ &H_j = \xi\, v_j. \end{aligned} \tag{3.14}$$

For C_l we have

$$\begin{aligned} &X_{\epsilon_i-\epsilon_j} = (-1)^{j-i}\xi\, v_{i\bar{j}} \qquad X_{-(\epsilon_i-\epsilon_j)} = \xi\, v_{j\bar{i}} \\ &X_{\epsilon_i+\epsilon_j} = -(-1)^{j-i}\xi\, v_{ij} \qquad X_{-(\epsilon_i+\epsilon_j)} = \xi\, v_{\overline{ji}} \\ &X_{2\epsilon_j} = -\xi\,(q+q^{-1})^{1/2} v_{jj} \qquad X_{-2\epsilon_j} = \xi\,(q+q^{-1})^{1/2} v_{\overline{jj}} \\ &H_i = (-1)^{l+1}\xi\, v_i. \end{aligned} \tag{3.15}$$

For D_l we have

$$\begin{aligned} &X_{\epsilon_i-\epsilon_j} = \xi\, v_{i\bar{j}} \qquad X_{-(\epsilon_i-\epsilon_j)} = \xi v_{j\bar{i}} \\ &X_{\epsilon_i+\epsilon_j} = \xi v_{ij} \qquad X_{-(\epsilon_i+\epsilon_j)} = \xi v_{\overline{ji}} \\ &H_i = \xi\, v_i \qquad \text{for } i < l \end{aligned} \tag{3.16}$$

$$H_l = \xi\,(v_{l-1} + (q+q^{-1})v_l). \tag{3.17}$$

Here $i < j \leqslant l$, $\bar{i} = 2l - i + 2$ for B_l and $\bar{i} = 2l - i + 1$ for C_l and D_l.

We could at this point give the structure constants. However, some of the structure constants are just too complicated. We would like to present the structure constants as concisely as possible. It turns out that with the introduction of the Killing form on our quantum Lie algebras, the unwieldy structure constants simplify immensely.

4. The Killing form

In this section we introduce an invariant Killing form on every quantum Lie algebra. We make the observation that it is an intertwiner for adjoint $\otimes$ adjoint $\rightarrow$ singlet, where by singlet we mean the trivial one-dimensional representation V^0. This allows us to calculate its values and we discover that they are simple q-deformation of the classical ones. To begin, we define the quantum analogue to the classical Killing form as follows.

Definition 4.1. The *quantum Killing form* is the map $\mathfrak{B} : \mathfrak{L}_h(\mathfrak{g})\hat{\otimes}\mathfrak{L}_h(\mathfrak{g}) \rightarrow \mathbb{C}[[h]]$ given by

$$\mathfrak{B}(a, b) = \mathrm{Tr}_\Psi(a\, b\, u). \tag{4.1}$$

Here u is the the element of $U_h(\mathfrak{g})$ satisfying the properties $u\, a\, u^{-1} = S^2(a) \forall a \in U_h(\mathfrak{g})$ and $\Delta(u) = u \otimes u$. The Tr_Ψ denotes the trace over the adjoint representation. This definition for the quantum Killing form reduces to that of the classical Killing form in the classical limit. It obviously exists and is non-degenerate because degeneracy would spoil the non-degeneracy of the classical Killing form. The definition is motivated by the following proposition.

Proposition 4.1. The Killing form is ad-invariant, i.e.

$$\mathfrak{B}([a, b]_h, c) = \mathfrak{B}(a, [b, c]_h) \qquad \forall a, b, c \in \mathfrak{g}_h. \tag{4.2}$$

We will prove this at the end of this section.

The following proposition, whose proof is trivial, informs us that unlike the classical Killing form, the quantum Killing form is not symmetric.

Proposition 4.2. The quantum Killing form $\mathfrak{B}$ is a non-degenerate, bilinear, nonsymmetric form. Symmetry is replaced by the relation

$$\mathfrak{B}(a, b) = \mathfrak{B}(b, S^2(a)). \tag{4.3}$$

The square of the antipode S^2 acts on the basis elements of $\mathfrak{L}_h(\mathfrak{g})$ by multiplication by a power of q. Therefore $S^2(\mathfrak{L}_h(\mathfrak{g})) \subset \mathfrak{L}_h(\mathfrak{g})$ and (4.3) makes sense.

The calculation of the Killing form is made simple when we realize that the Killing form is an intertwiner for adjoint $\otimes$ adjoint $\to$ singlet.

Proposition 4.3. The Killing form defined in (4.1) is an intertwiner from adjoint$\otimes$adjoint $\to$ singlet, i.e.

$$\mathfrak{B}([c_{(1)}, a]_h, [c_{(2)}, b]_h) = \epsilon(c)\mathfrak{B}(a, b) \qquad \forall a, b, c \in \mathfrak{L}_h(\mathfrak{g}). \tag{4.4}$$

Proof. From the definition of the Killing form and Lie bracket, the left-hand side becomes

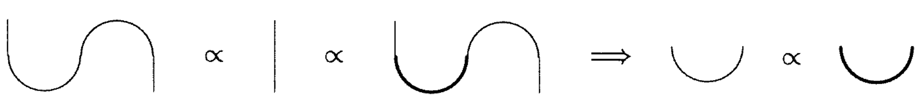
$$\begin{aligned} \mathrm{Tr}_\Psi(c_{(1)}aS(c_{(2)})c_{(3)}bS(c_{(4)})u) &= \mathrm{Tr}_\Psi(c_{(1)}abS(c_{(2)})u) \\ &= \mathrm{Tr}_\Psi(c_{(1)}abuS^{-1}(c_{(2)})) \\ &= \mathrm{Tr}_\Psi(S^{-1}(c_{(2)})c_{(1)}abu) \qquad \text{by cyclicity} \\ &= \mathrm{Tr}_\Psi(\epsilon(c)abu). \end{aligned}$$

□

Such an intertwiner is unique up to rescaling. To see this let $h, g : V^\Psi \otimes V^\Psi \to V^0$ be any two intertwiners. We represent them by light resp. heavy curves as follows

(4.5)

We can compose either h or g with h^{-1} to obtain two intertwiners from V^Ψ to itself.



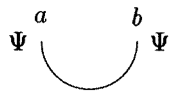
(4.6)

By Schur's lemma they both have to be proportional to the identity. Thus, h and g have to be proportional.

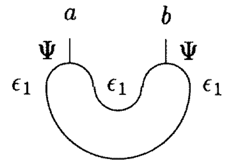
One can also define a form on all of $U_h(\mathfrak{g})$ by equation (4.1). This gives the Rosso form [13] which is the unique form on $U_h(\mathfrak{g})$ for which (4.4) is valid for any $a, b, c \in U_h(\mathfrak{g})$.

We calculate an intertwiner B from adjoint $\otimes$ adjoint $\rightarrow$ singlet using the Clebsch–Gordan coefficients from section 3.3 according to the formula

$$B(a, b) =$$



$$= \frac{1}{N}$$



(4.7)

where $N = (q + q^{-1})^3$ for B_l and D_l, and $N = -(q + q^{-1})^3$ for C_l. By the above propositions B is proportional to the quantum Killing form $\mathfrak{B}$ defined in (4.1).

The Killing form on the roots has the simple form

$$B(X_\alpha, X_\beta) = q^{-\rho\cdot\alpha}\delta_{\alpha,-\beta} \qquad \forall\alpha \in R. \tag{4.8}$$

The Killing form on the Cartan subalgebra for the algebras A_l, B_l, C_l and D_l respectively, is given by

$$(A_l)_h : \qquad B(H_i, H_j) = \begin{pmatrix} [2]_q & -1 & & & & \\ -1 & [2]_q & -1 & & & \\ & & & & & \\ & & & & & \\ & & & -1 & [2]_q & -1 \\ & & & & -1 & [2]_q \end{pmatrix} \tag{4.9}$$

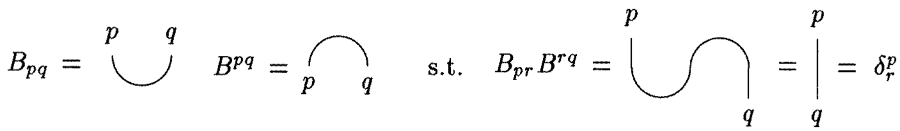
$$(B_l)_h : \qquad B(H_i, H_j) = \begin{pmatrix} [2]_q & -1 & & & & \\ -1 & [2]_q & -1 & & & \\ & & & & & \\ & & & & & \\ & & & -1 & [2]_q & -1 \\ & & & & -1 & 1 \end{pmatrix} \tag{4.10}$$

$$(C_l)_h : \qquad B(H_i, H_j) = \begin{pmatrix} [2]_q & -1 & & & & \\ -1 & [2]_q & -1 & & & \\ & & & & & \\ & & & & & \\ & & & -1 & [2]_q & -1 \\ & & & & -1 & \frac{q^2+q^{-2}}{q+q^{-1}} \end{pmatrix} \tag{4.11}$$

$$(D_l)_h : \qquad B(H_i, H_j) = \begin{pmatrix} [2]_q & -1 & & & & & \\ -1 & [2]_q & -1 & & & & \\ & & & & & & \\ & & & & & & \\ & & & -1 & [2]_q & -1 & -1 \\ & & & & -1 & [2]_q & 0 \\ & & & & -1 & 0 & [2]_q \end{pmatrix}. \tag{4.12}$$

These expressions are surprisingly simple q-deformations of the classical ones. In particular $B(H_i, H_j) \neq 0$ if and only if $\alpha_i \cdot \alpha_j \neq 0$. The Killing form for $(A_l)_h$ was determined in [5].

We introduce the notation $B(H_i, H_j) = B_{ij}$, $B(X_\alpha, X_{-\alpha}) = B_{\alpha,-\alpha}$. We define B^{ij} such that $B_{ij}B^{jk} = \delta_i^k$ and $B^{-\alpha,\alpha}$ such that $B_{\alpha,-\alpha}B^{-\alpha,\alpha} = 1$, or, equivalently, introducing composite indices $p, q, r = \{i, \alpha\}$,



. (4.13)

We give the formulae for B^{ij} with $i \leqslant j$, the remaining ones are obtained by symmetry, $B^{ji} = B^{ij}$.

$$(A_l)_h : B^{ij} = [i]_q \frac{q^{l-j+1} - q^{j-l-1}}{q^{l+1} - q^{-l-1}} \tag{4.14}$$

$$(B_l)_h : B^{ij} = [i]_q \frac{q^{l-j-1/2} + q^{j-l+1/2}}{q^{l-1/2} + q^{-l+1/2}} \tag{4.15}$$

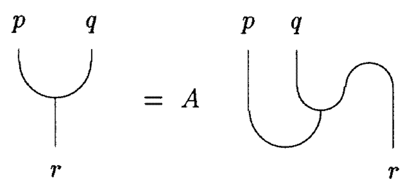
$$(C_l)_h : B^{ij} = [i]_q \frac{q^{l-j+1} + q^{j-l-1}}{q^{l+1} + q^{-l-1}} \tag{4.16}$$

$$\begin{aligned} (D_l)_h : B^{ij} &= [i]_q \frac{q^{l-j-1} + q^{j-l+1}}{q^{l-1} + q^{-l+1}} \qquad i < l-1,\ j < l \\ B^{l-1,l-1} &= B^{l,l} = [l]_q \frac{(q+q^{-1})^{-1}}{q^{l-1} + q^{-l+1}} \\ B^{l-1,l} &= [l-2]_q \frac{(q+q^{-1})^{-1}}{q^{l-1} + q^{-l+1}} \\ B^{i,l} &= B^{i,l-1}. \end{aligned} \tag{4.17}$$

We now give the proof of proposition 4.1, i.e. we will show that

$$B([a, b]_h, c) = B(a, [b, c]_h) \qquad \forall a, b, c \in \mathfrak{g}_h. \tag{4.18}$$

Proof. Because the intertwiner from adjoint to adjoint $\otimes$ adjoint is unique up to rescaling we have



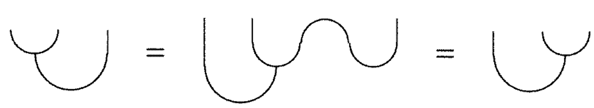
. (4.19)

The constant of proportionality $A \in \mathbb{C}[[h]]$ can be determined by setting $p = r = X_\alpha$ and $q = H_i$. Doing this we obtain the relation

$$-r_\alpha = A B_{\alpha,-\alpha} l_{-\alpha}(H_i) B^{-\alpha,\alpha}.$$

Using $B_{\alpha,-\alpha}B^{-\alpha,\alpha} = 1$ and the relation $l_{-\alpha} = -r_\alpha$ from (2.8) and (2.9) we see that $A = 1$.

We now write the left-hand side of (4.18) in graphical form and use the above identity (4.19) to manipulate the diagram to the right-hand side of (4.18).



. (4.20)

□

Besides the Killing form there exists another natural bilinear form $\langle , \rangle$ on the quantum Lie algebras. This is the unique (up to rescaling) non-degenerate bilinear form which satisfies

$$\langle [a,b], c\rangle = \langle b, [a^\dagger, c]\rangle \qquad \forall a, b, c \in \mathfrak{L}_h(\mathfrak{g}) \tag{4.21}$$

where † denotes the algebra antiautomorphism of $U_h(\mathfrak{g})$ defined by

$$(x_i^\pm)^\dagger = x_i^\mp \qquad h_i^\dagger = h_i. \tag{4.22}$$

(If we define † to be antilinear then the form $\langle , \rangle$ has to be sesquilinear.) On the quantum Lie algebra generators it acts as

$$X_\alpha^\dagger = q^{-\rho\cdot\alpha} X_{-\alpha} \qquad H_i^\dagger = H_i. \tag{4.23}$$

The form $\langle , \rangle$ is symmetric and is given by

$$\langle X_\alpha, X_\beta\rangle = \delta_{\alpha\beta} \qquad \langle H_i, H_j\rangle = B_{ij} \qquad \langle X_\alpha, H_i\rangle = \langle H_i, X_\alpha\rangle = 0. \tag{4.24}$$

5. The quantum structure constants

We only give the minimal set of the structure constants. The remaining ones can be calculated using the symmetry properties, see equations (2.8)–(2.12). Thus, we give only the left quantum roots l_α for positive α. From these, those for negative α as well as the r_α and ${g_\alpha}^k$ can be obtained using equations (2.8), (2.9) and (2.12). The structure constants ${f_{ij}}^k$ for the Cartan subalgebra simplify dramatically if one lowers the last index using the Killing form, i.e. $f_{ijk} := {f_{ij}}^m B(H_m, H_k)$. Since the Killing form is symmetric on the Cartan subalgebra it is easy to see, using (2.8), (2.9) and (2.11), that f_{ijk} is completely symmetric. We also observed that $f_{ijk} \neq 0$ iff $\alpha_i \cdot \alpha_j \neq 0$. This implies in particular that the structure constants f_{ijk} are non-zero only if at least two of the indices are the same. (This is so because f_{ijk} is only non-zero if f_{ijk}, f_{ikj} and f_{jki} are non-zero. In other words, f_{ijk} is non-zero if there exist three simple roots such that $\alpha_i \cdot \alpha_j$, $\alpha_i \cdot \alpha_k$ and $\alpha_j \cdot \alpha_k$ are non-zero. There exist no three distinct simple roots that satisfy this requirement.) So below we give only a few f's, all others can be obtained by symmetry or are zero. We give only enough of the $N_{\alpha,\beta}$ so that the others can be obtained using (2.8)–(2.10).

5.1. The quantum Lie algebra $(B_l)_h$

The quantum roots are

$$\begin{aligned}
l_{\epsilon_j-\epsilon_k}(H_i) &= q^{l-i-1/2}\delta_{ij} - q^{i-l+1/2}\delta_{ik} - q^{l-i-5/2}\delta_{i+1,j} + q^{i-l+5/2}\delta_{i+1,k}\\
l_{\epsilon_j+\epsilon_k}(H_i) &= q^{l-i-1/2}\delta_{ij} + q^{l-i+3/2}\delta_{ik} - q^{l-i-5/2}\delta_{i+1,j} - q^{l-i-1/2}\delta_{i+1,k}\\
l_{\epsilon_k}(H_i) &= q^{l-i-1/2}\delta_{ik} - q^{l-i-5/2}\delta_{i+1,k} + (q^{3/2} - q^{1/2})\delta_{il}
\end{aligned} \tag{5.1}$$

where $j < k \leqslant l$, $i \leqslant l$. The structure constants for the Cartan subalgebra are

$$\begin{aligned}
f_{iii} &= (q^{l-i-3/2} + q^{i-l+3/2})(q^2 - q^{-2})\\
f_{lll} &= (q + q^{-1})(q^{1/2} - q^{-1/2}) - (q^{1/2} - q^{-1/2})\\
f_{i\pm1,i\pm1,i} &= \mp(q^{l-i-3/2} - q^{i-l+3/2}).
\end{aligned} \tag{5.2}$$

The remaining structure constants are determined by

$$\begin{aligned}
N_{\epsilon_i-\epsilon_j,\epsilon_k-\epsilon_m} &= q^{l-k-1}\delta_{jk} - q^{m-l+1}\delta_{im} \\
N_{\epsilon_i-\epsilon_j,\epsilon_k+\epsilon_m} &= q^{l-k-1}\delta_{jk} + (q^{l-m+1}\delta_{i>k} - q^{l-m}\delta_{i<k})\delta_{jm} \\
N_{\epsilon_i-\epsilon_j,\epsilon_m} &= q^{l-m-1}\delta_{jm} \\
N_{\epsilon_j,\epsilon_m} &= q^{1/2}\delta_{j>m} - q^{-1/2}\delta_{j<m}
\end{aligned} \tag{5.3}$$

where $i < j \leqslant l$ and $k < m \leqslant l$.

5.2. *The quantum Lie algebra $(C_l)_h$*

The quantum roots are

$$\begin{aligned}
l_{\epsilon_j-\epsilon_k}(H_i) &= q^{l-i+3}\delta_{ij} - q^{i-l-3}\delta_{ik} - q^{l-i+1}\delta_{i+1,j} + q^{i-l-1}\delta_{i+1,k} \\
l_{\epsilon_j+\epsilon_k}(H_i) &= q^{l-i+3}\delta_{ij} + q^{l-i+1}\delta_{ik} - q^{l-i+1}\delta_{i+1,j} - q^{l-i-1}\delta_{i+1,k} \\
l_{2\epsilon_k}(H_i) &= (q+q^{-1})(q^{l-i+2}\delta_{ik} - q^{l-i}\delta_{i+1,k})
\end{aligned} \tag{5.4}$$

where $j < k \leqslant l$, $i \leqslant l$. The structure constants for the Cartan subalgebra are

$$\begin{aligned}
f_{iii} &= (q^2 - q^{-2})(q^{l-i+2} + q^{i-l-2}) \\
f_{i\pm1,i\pm1,i} &= \mp(q^{l-i+2} - q^{i-l-2}).
\end{aligned} \tag{5.5}$$

The remaining structure constants are determined by

$$\begin{aligned}
N_{\epsilon_i-\epsilon_j,\epsilon_k-\epsilon_m} &= (-1)^{l-i}q^{i-l-5/2}\delta_{im} - (-1)^{l-j}q^{l-j+5/2}\delta_{jk} \\
N_{\epsilon_i-\epsilon_j,\epsilon_k+\epsilon_m} &= -(-1)^{l-j}\{q^{l-j+5/2}\delta_{jk} + (q^{l-j+3/2}\delta_{i<k} \\
&\quad +(q+q^{-1})^{1/2}q^{l-j+1}\delta_{ik} + q^{l-j+1/2}\delta_{i>k})\delta_{jm}\} \\
N_{\epsilon_i-\epsilon_j,2\epsilon_m} &= -(-1)^{l-j}(q+q^{-1})^{1/2}q^{l-j+2}\delta_{jm} \\
N_{\epsilon_i-\epsilon_j,-2\epsilon_m} &= (-1)^{l-j}(q+q^{-1})^{1/2}q^{i-l-1}\delta_{im}
\end{aligned} \tag{5.6}$$

where $i < j \leqslant l$ and $k < m \leqslant l$.

5.3. *The quantum Lie algebra $(D_l)_h$*

The quantum roots are

$$\begin{aligned}
l_{\epsilon_j-\epsilon_k}(H_i) &= (q^{l-i-1}\delta_{ij} - q^{i-l+1}\delta_{ik} - q^{l-i-3}\delta_{i+1,j} + q^{i-l+3}\delta_{i+1,k})\delta_{i<l} \\
&\quad +(-\delta_{ik} + \delta_{i-1,j} - \delta_{i-1,k})\delta_{il} \\
l_{\epsilon_j+\epsilon_k}(H_i) &= (q^{l-i-1}\delta_{ij} + q^{l-i+1}\delta_{ik} - q^{l-i-3}\delta_{i+1,j} - q^{l-i-1}\delta_{i+1,k})\delta_{i<l} \\
&\quad +(q^2\delta_{ik} + \delta_{i-1,j} + q^2\delta_{i-1,k})\delta_{il}
\end{aligned} \tag{5.7}$$

where $j < k \leqslant l$, $i \leqslant l$. The structure constants for the Cartan subalgebra are

$$\begin{aligned}
f_{iii} &= (q^2 - q^{-2})(q^{l-i-2} + q^{i-l+2}) \\
f_{j+1,j+1,j} &= -(q^{l-j-2} - q^{j-l+2}) \\
f_{i-1,i-1,i} &= +(q^{l-i-2} - q^{i-l+2})
\end{aligned} \tag{5.8}$$

where $i < l$, $j < l-1$. Because of the Dynkin diagram automorphism τ, which interchanges H_l and H_{l-1}, we do not need to give the structure constants involving H_l, they are equal to those involving H_{l-1}. The f's involving both l and $l-1$ are zero. The remaining structure constants are determined by

$$\begin{aligned} N_{\epsilon_i-\epsilon_j,\epsilon_k-\epsilon_m} &= q^{l-j-3/2}\delta_{jk} - q^{i-l+3/2}\delta_{im} \\ N_{\epsilon_i-\epsilon_j,\epsilon_k+\epsilon_m} &= q^{l-j-3/2}\delta_{jk} - (q^{l-j-1/2}\delta_{i<k} - q^{l-j+1/2}\delta_{i>k})\delta_{jm} \end{aligned} \tag{5.9}$$

where $i < j \leqslant l$ and $k < m \leqslant l$.

5.4. *The quantum Lie algebras $(A_l)_h$*

For completeness we also give the structure constants for the quantum Lie algebras associated to $\mathfrak{g} = A_l$ which were determined by a different method in [5], see also [6]. There is a family of quantum Lie algebras $(A_l)_h(\chi)$ depending on a parameter χ. This is due to the fact that in the case of A_l, the adjoint representation appears in adjoint $\otimes$ adjoint with multiplicity 2. The parameter χ can be written as a fraction $\chi = s/t$ with $s, t \in \mathbb{C}[[h]]$ and with the restriction that $(s+t)^{-1} \in \mathbb{C}[[h]]$. A_l is isomorphic to sl_{l+1}.

The quantum roots are

$$l_{\epsilon_j-\epsilon_k}(H_i) = (q^{1-i}\delta_{ij} - q^{-1-i}\delta_{i+1,j})(s + t\,q^{l+1}) - (q^{i-1}\delta_{ik} - q^{i+1}\delta_{i+1,k})(s + t\,q^{-l-1}) \tag{5.10}$$

where $j \neq k \leqslant l+1$, $i \leqslant l$. The structure constants for the Cartan subalgebra are

$$\begin{aligned} f_{iii} &= s(q^2 - q^{-2})(q^{-i} + q^i) + t(q^2 - q^{-2})(q^{l-i+1} + q^{i-l-1}) \\ f_{i\pm1,i\pm1,i} &= \mp s(q^{-i} - q^i) \mp t(q^{l-i+1} - q^{i-l-1}) \end{aligned} \tag{5.11}$$

where $i \leqslant l$. Finally,

$$N_{\epsilon_i-\epsilon_j,\epsilon_k-\epsilon_m} = q^{1/2-j}(s + t\,q^{l+1})\delta_{jk} - q^{i-1/2}(s + t\,q^{-l-1})\delta_{im} \tag{5.12}$$

where $i \neq j \leqslant l+1$ and $k \neq m \leqslant l+1$.

5.5. *The structure of the Cartan subalgebra*

One of the novel features of quantum Lie algebras is the non-vanishing of the quantum Lie bracket between elements of the Cartan subalgebra. Our explicit results for the corresponding structure constants f_{ijk} given above have lead us to the following observation:

$$f_{iij} = B_{ij}(l_{\alpha_j}(H_i) - r_{\alpha_j}(H_i)). \tag{5.13}$$

In other words, the Lie brackets of the Cartan subalgebra elements are given by the amount of split between left and right quantum roots. In the following we will abbreviate this q-antisymmetric combination of left and right roots by $a_\alpha := l_\alpha - r_\alpha$. The Lie bracket relations are then

$$[H_i, H_j]_h = B_{ij}(a_{\alpha_i}(H_j)H^j + a_{\alpha_j}(H_i)H^i). \tag{5.14}$$

5.6. *The quantum root space*

The quantum root space $\mathcal{H}^*$ is the dual space to the Cartan subalgebra, i.e. it is the space of linear functionals on $\mathcal{H}$ with values in $\mathbb{C}[[h]]$. The left and right quantum roots l_α and r_α are particular elements of $\mathcal{H}^*$.

The Killing form B on $\mathcal{H}$ provides a natural pairing between elements $H \in \mathcal{H}$ and linear functionals $v_H \in \mathcal{H}^*$ defined by

$$v_H(H') := B(H, H') \qquad \forall H' \in \mathcal{H}. \tag{5.15}$$

Let $v_i := v_{H_i}$ be the elements of $\mathcal{H}^*$ dually paired with the generators H_i of $\mathcal{H}$. Then $\{v_i\}_{i=1,\ldots,\mathrm{rank}\mathfrak{g}}$ is a basis for $\mathcal{H}^*$. With our choice of the H_i the v_i are proportional to the q-symmetric combination of the simple left and right quantum roots,

$$v_i(H_j) := B(H_i, H_j) = \frac{1}{\xi_i}(l_{\alpha_i}(H_j) + r_{\alpha_i}(H_j)). \tag{5.16}$$

The factors of proportionality ξ_i are given by

$$B_l : \xi_i = q^{i-l+3/2} + q^{l-i-3/2} \tag{5.17}$$

$$C_l : \xi_i = q^{i-l-2} + q^{l-i+2} \qquad \text{for } i < l \qquad \xi_l = (q + q^{-1})^2 \tag{5.18}$$

$$D_l : \xi_i = q^{i-l+2} + q^{l-i-2} \qquad \text{for } i < l \qquad \xi_l = \xi_{l-1} = (q + q^{-1}). \tag{5.19}$$

These were chosen so as to make the structure constants as simple as possible.

There is a natural inner product on $\mathcal{H}^*$ given by

$$\langle v_H, v_{H'}\rangle := B(H, H'). \tag{5.20}$$

On our basis this gives $\langle v_i, v_j\rangle = B_{ij}$. Thus, we have chosen our basis vectors to all have length squared equal to $[2]_q = (q + q^{-1})$ except for v_l for B_l and C_l. The simple quantum roots l_{α_i} on the other hand, all have different lengths, quite unlike the classical simple roots.

6. Other relations between the structure constants

We have already observed that the product of any number of intertwiners is still an intertwiner. This observation is indeed a very powerful one. Below we will use it to derive some interesting relations between the structure constants.

The calculations leading to our results for the structure constants presented in the previous section were rather lengthy. It is therefore very important to have powerful checks on the correctness of the results. The structure constants in the classical limit have been compared with the classical structure constants and they agree. We have also verified the symmetry relations in section 2.1 for the quantum structure constants. The relations derived in this section provide further checks. For quantum Lie algebras of low rank we have checked that these relations are satisfied by our results. We have made the Mathematica notebooks containing these calculations available on the Internet at http://www.mth.kcl.ac.uk/∼delius/q-lie/

In the following we use the intertwiners from adjoint ⊗ adjoint → adjoint and singlet → adjoint ⊗ adjoint. This second intertwiner is the inverse of the Killing form. Adopting the same notation as in section 4 we denote this intertwiner by B^{pq} and define it so that $B_{pr}B^{rq} = \delta_p^q$. Thus in particular $B^{\alpha,-\alpha} = q^{-\rho\cdot\alpha}$. B^{ij} (given in equations (4.14)–(4.17)) can be used to raise the indices which are lowered with B_{ij}.

We come now to the derivation of two new relations. They are obtained by constructing intertwiners singlet → adjoint and adjoint → adjoint. Further relations can easily be derived by the same method.

6.1. *The intertwiner singlet → adjoint*

An intertwiner from the singlet to the adjoint is given in figure 2(a). This intertwiner should be zero and so we immediately arrive at a relation between the structure constants. Setting $p = k$ we have

$$\sum_\alpha -{g_\alpha}^k B^{\alpha,-\alpha} + f_{ij}^k B^{ij} = 0. \tag{6.1}$$

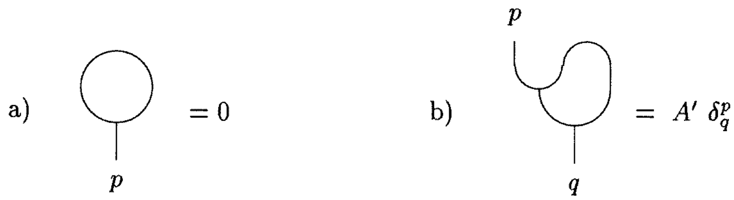


Figure 2. Graphical representation of the relations. (a) There is no non-zero intertwiner from singlet to adjoint. (b) An intertwiner from adjoint to adjoint is proportional to the identity.

We multiply this equation by B_{km} and sum over k. We use $B_{\alpha,-\alpha} = B^{\alpha,-\alpha} = q^{-\rho\cdot\alpha}$ and relation (2.12) between ${g_\alpha}^k$ and the quantum roots to obtain

$$\sum_\alpha l_\alpha(H_m)q^{-2\rho\cdot\alpha} + \sum_i {f_{mi}}^i = 0. \tag{6.2}$$

6.2. The intertwiner adjoint → adjoint

The intertwiner from adjoint to adjoint is given in figure 2(b). This intertwiner should be proportional to the identity map. Setting $p = q = \alpha$ we have

$$\sum_\beta N_{\alpha,\beta}N_{\alpha+\beta,-\beta}B^{\beta,-\beta} + r_\alpha(H_i)r_\alpha(H_j)B^{ij} - {g_\alpha}^i l_\alpha(H_i)B^{-\alpha,\alpha} = A'. \tag{6.3}$$

Using once again relation (2.12) between ${g_\alpha}^i$ and the quantum roots, the relations between the $N_{\alpha\beta}$, and the value of $B^{\alpha,-\alpha}$ the above equations become

$$\sum_\beta (N_{\alpha,\beta})^2 q^{-2\rho\cdot\beta} + r_\alpha(H_i)r_\alpha(H_j)B^{ij} + l_\alpha(H_i)l_\alpha(H_j)B^{ij} = A' \qquad \forall\alpha\in R. \tag{6.4}$$

Setting $p = i$ and $q = j$ in figure 2(b) we obtain

$$-\sum_\alpha l_\alpha(H_i)B^{\alpha,-\alpha}\,{g_\alpha}^j + {f_{ik}}^m B^{kl}{f_{ml}}^j = A'\delta_i^j \tag{6.5}$$

which can be re-expressed as

$$\sum_\alpha l_\alpha(H_i)l_\alpha(H_j) + {f_i}^{ml}f_{mlj} = A'B_{ij}. \tag{6.6}$$

7. Discussion

We have shown how to calculate the structure constants of the quantum Lie algebras associated to B_l, C_l and D_l. These calculations were rendered manageable by the observation that the quantum structure constants are just the inverse Clebsch–Gordan coefficients for adjoint ⊗ adjoint → adjoint. The structure constants satisfy the symmetries discovered in [4]. We have introduced an ad-invariant Killing form and shown that it is proportional to the intertwiner from adjoint ⊗ adjoint → singlet. Because the composition of intertwiners is also an intertwiner we were able to calculate many intertwiners indirectly. For example we calculated the Killing form by building the intertwiner from adjoint ⊗ adjoint → singlet in terms of the intertwiners from vector ⊗ vector into adjoint and singlet. This meant that we did not have to evaluate the usual trace over the adjoint representation.

As is well known, the structure constants of the simple complex Lie algebras are determined entirely in terms of their simple roots. Eventually we would hope to arrive at a similar result for the quantum Lie algebras. In this paper we have come one step closer to this goal by our observation that the structure constants f_{ijk} of the Cartan subalgebra are completely determined in terms of the quantum roots according to equation (5.13). The Killing form B_{ij} is expressed in terms of the l_α by equation (5.16). Already in [4] it was found that the left quantum roots l_α for positive α are enough to determine those for negative α by equation (2.9), the r_α by (2.8) and the $g_\alpha{}^k$ by equation (2.12). Thus now all quantum Lie bracket relations are determined by the left quantum roots l_α for positive α and the $N_{\alpha\beta}$. What is still missing is a deeper understanding of the $N_{\alpha\beta}$ and of how to obtain the higher quantum roots from the simple quantum roots. For recent progress see [11].

The expressions for the quantum roots in equations (5.1), (5.4) and (5.7) are unexpectedly simple. If one writes the classical expressions for the roots in the same form, one notices that generically the quantum expressions are obtained from these by replacing every 1 by a power of q and every 2 by $(q+q^{-1})$ times a power of q. Thus, in particular $l_\alpha(H_i) \neq 0$ if and only if classically $\alpha(H_i) \neq 0$. There is, however, one exception to this simplicity: in $(B_l)_h$ we have found that $l_{\epsilon_k}(H_l) \neq 0$ also for $k < l-1$.

Also the matrices (4.10)–(4.12) describing the quantum Killing form on the Cartan subalgebras are surprisingly simple. In particular we find that only those entries in the matrices are non-zero which are also non-zero classically.

There still remain a lot of unanswered questions. In particular: What is a good axiomatic setting for the theory of quantum Lie algebras. How should one q-deform the Jacobi identity? What characterizes the quantum root system? What are q-Weyl reflections? How does one define representations of these non-associative algebras? And many more. For a more complete bibliography and more recent results on quantum Lie algebras see http://www.mth.kcl.ac.uk/˜delius/q-lie/

Acknowledgment

CG was financially supported by EPSRC.

Appendix A. Quantized enveloping algebras

For an introduction to Lie algebras consult [14, 12] and for quantized enveloping algebras consult [3].

Definition A.1. Let $\mathfrak{g}$ be a finite-dimensional simple complex Lie algebra with symmetrizable Cartan matrix a_{ij}. The *quantized enveloping algebra* $U_h(\mathfrak{g})$ is the unital associative algebra over $\mathbb{C}[[h]]$ (completed in the h-adic topology) with generators x_i^+, x_i^-, h_i, $1 \leqslant i \leqslant \mathrm{rank}(\mathfrak{g})$ and relations

$$\begin{gathered} h_i h_j = h_j h_i \qquad h_i x_j^{\pm} - x_j^{\pm} h_i = \pm a_{ij} x_j^{\pm} \\ x_i^+ x_j^- - x_j^- x_i^+ = \delta_{ij} \frac{q_i^{h_i} - q_i^{-h_i}}{q_i - q_i^{-1}} \\ \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} (x_i^{\pm})^k x_j^{\pm} (x_i^{\pm})^{1-a_{ij}-k} = 0 \qquad i \neq j. \end{gathered} \tag{A.1}$$

Here $\begin{bmatrix} a \\ b \end{bmatrix}_q$ are the q-binomial coefficients.

We have defined $q_i = \mathrm{e}^{d_i h}$ where the d_i are chosen so that $d_i a_{ij}$ is a symmetric matrix. We choose $d_i = \alpha_i^2/2$ where the simple roots are as given at the beginning of section 3. An alternative convention is to choose d_i to be coprime integers. In the case of the algebra B_l these two conventions differ and our conventions lead to $d_i = 1$ for $i = 1, \ldots, l-1$ and $d_l = \frac{1}{2}$. The Cartan matrix is defined to be $a_{ij} = 2\alpha_i \cdot \alpha_j / \alpha_i^2$.

The Hopf algebra structure of $U_h(\mathfrak{g})$ is given by the comultiplication $\Delta : U_h(\mathfrak{g}) \to U_h(\mathfrak{g}) \hat{\otimes} U_h(\mathfrak{g})$ defined by

$$\Delta(h_i) = h_i \hat{\otimes} 1 + 1 \hat{\otimes} h_i \tag{A.2}$$

$$\Delta(x_i^{\pm}) = x_i^{\pm} \hat{\otimes} q_i^{-h_i/2} + q_i^{h_i/2} \hat{\otimes} x_i^{\pm} \tag{A.3}$$

and the antipode S and co-unit ϵ defined by

$$S(h_i) = -h_i \qquad S(x_i^{\pm}) = -q_i^{\mp 1} x_i^{\pm} \qquad \epsilon(h_i) = \epsilon(x_i^{\pm}) = 0. \tag{A.4}$$

Definition A.2. The Cartan involution $\theta : U_h(\mathfrak{g}) \to U_h(\mathfrak{g})$ is given by the formulae

$$\theta(x_i^{\pm}) = -x_i^{\mp} \qquad \theta(h_i) = -h_i. \tag{A.5}$$

It is a Hopf-algebra isomorphism: $U_h(\mathfrak{g}) \to U_h^{\mathrm{op}}(\mathfrak{g})$ where $U_h^{\mathrm{op}}(\mathfrak{g})$ is the opposite Hopf algebra, whose Hopf structure is described by the opposite coproduct Δ^{op} and the inverse antipode S^{-1}.

Definition A.3. q-conjugation $\sim$: $\mathbb{C}[[h]] \to \mathbb{C}[[h]]$, $a \mapsto \tilde{a}$ is the $\mathbb{C}$-linear ring automorphism defined by $\tilde{h} = -h$.

Appendix B. Modified Schur's lemma

Lemma B.1 (Schur's lemma). Let $V[[h]]$ and $W[[h]]$ be two finite-dimensional indecomposable $U_h(\mathfrak{g})$-modules and let $f, g : V[[h]] \to W[[h]]$ be two $U_h(\mathfrak{g})$-module homomorphism. Then:

(1) if $f(\mathrm{mod}\, h) \neq 0$ then f is an isomorphism;

(2) $\exists c \in \mathbb{C}[[h]]$ such that $f = cg$ or $g = cf$.

Proof. (1) $\mathrm{Ker}(f)$ is a submodule of $V[[h]]$. However, $V[[h]]$ is an indecomposable $U_h(\mathfrak{g})$-module and so $\mathrm{Ker}(f)$ must be of the form $cV[[h]]$ for some non-invertible $c \in \mathbb{C}[[h]]$. However, if $\mathrm{Ker}(f)$ has this form then $f(cx) = cf(x) = 0 \forall x \in V[[h]]$, i.e. $\mathrm{Ker}(f) = V[[h]]$. Therefore $\mathrm{Ker}(f) = 0$. Equally, $\mathrm{Im}\,(f)$ is a submodule of $W[[h]]$ and so (2) Let v_0 and w_0 be the highest weight states in V and W. Then because f, g are $U_h(\mathfrak{g})$-homomorphisms $f(v_0)$ and $g(v_0)$ must also be highest weight states in $W[[h]]$, that is $\exists c_1, c_2 \in \mathbb{C}[[h]]$ such that $f(v_0) = c_2 w_0$ and $g(v_0) = c_1 w_0$. Then $(c_1 f - c_2 g)(v_0) = 0$ which means that $c_1 f - c_2 g$ is not an isomorphism and so by the first part of the lemma $c_1 f - c_2 g = 0(\mathrm{mod}\, h)$. By the same argument $h^{-1}(c_1 f - c_2 g) = 0$, etc. Hence, $c_1 f = c_2 g$. □

Appendix C. Clebsch–Gordan coefficients

Let (π^{μ}, V^{μ}) and (π^{ν}, V^{ν}) be two indecomposable $U_h(\mathfrak{g})$-modules. Consider an indecomposable $U_h(\mathfrak{g})$-module $(\pi^{\lambda}, V^{\lambda})$ homomorphically embedded in $V^{\mu} \otimes V^{\nu}$. As a basis for V^{λ} we choose $\{v_{\lambda}^{a}\}$. So we have for the action of $U_h(\mathfrak{g})$ on V^{λ}

$$\pi^{\lambda}(x) v_{\lambda}^{c} = v_{\lambda}^{d} \pi_{d}^{\lambda c}(x) = \pi_{d}^{\lambda c}(x) C_{a'b'}^{\mu\nu}\big|_{\lambda}^{d} v_{\mu}^{a'} \otimes v_{\nu}^{b'} \tag{C.1}$$

where $\{v_\mu^{a'} \otimes v_\nu^{b'}\}$ is the natural basis on $V^\mu \otimes V^\nu$ and $C^{\mu\nu}_{a'b'}|^d_\lambda$ are the Clebsch–Gordan coefficients describing the embedding $V^\lambda \to V^\mu \otimes V^\nu$. The action of $U_h(\mathfrak{g})$ on $V^\mu \otimes V^\nu$ is defined using the coproduct, i.e.

$$\begin{aligned}\pi^\lambda(x)v^c_\lambda &= C^{\mu\nu}_{ab}|^c_\lambda(\pi^\mu \otimes \pi^\nu)(\Delta(x))(v^a_\mu \otimes v^b_\nu)\\ &= C^{\mu\nu}_{ab}|^c_\lambda(v^{a'}_\mu \otimes v^{b'}_\nu)(\pi^{\mu a}_{a'} \otimes \pi^{\nu b}_{b'})(\Delta(x)).\end{aligned} \tag{C.2}$$

These two actions of $U_h(\mathfrak{g})$ coincide and so the Clebsch–Gordan coefficients satisfy the intertwiner property

$$\pi^\lambda(x)^c_d C^{\mu\nu}_{a'b'}|^d_\lambda = C^{\mu\nu}_{ab}|^c_\lambda(\pi^{\mu a}_{a'} \otimes \pi^{\nu b}_{b'})(\Delta(x)). \tag{C.3}$$

Appendix D. The $(C_2)_h$ algebra

In this appendix we compare our results for $(C_2)_h$ with the results given in [4]. The Cartan matrix for C_2 is

$$\begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix} \tag{D.1}$$

and the positive roots are $\alpha_1, \alpha_2, \alpha_1+\alpha_2$, and $2\alpha_1+\alpha_2$. The left quantum roots are given by

$$\begin{aligned}&l_{\alpha_1}(H_1, H_2) = ((q+q^{-1})(q^2-1+q^{-2})q, -q^{-3}))\\ &l_{\alpha_2}(H_1, H_2) = (-(q+q^{-1})q, q^2(q+q^{-1}))\\ &l_{\alpha_1+\alpha_2}(H_1, H_2) = ((q+q^{-1})(q-q^{-1})q^2, q))\\ &l_{2\alpha_1+\alpha_2}(H_1, H_2) = ((q+q^{-1})q^3, 0).\end{aligned} \tag{D.2}$$

We agree with the previous results in [4] if we first q-conjugate our results and then make the following transformations:

$$\begin{aligned}&H_1 \longrightarrow l\frac{(q^{-2}+q^2-1)}{q+q^{-1}}h_1\\ &H_2 \longrightarrow lh_2\end{aligned} \tag{D.3}$$

where l is as defined in [4] and h_1, h_2 are the Cartan subalgebra generators used in [4].

In table D.1 we give $N_{\alpha,\beta}/(q+q^{-1})^{1/2}$ for α positive. The rows are labelled by α and the columns are labelled by β. The $N_{-\alpha,\beta}$ are equal to $-\tilde{N}_{\alpha,-\beta}$.

We get agreement with [4] if we first q-conjugate and then make the following transformations:

$$\begin{aligned}&X_{\pm\alpha_1} \longrightarrow \mp\xi X_{\pm\alpha_1}\\ &X_{\pm\alpha_2} \longrightarrow \mp\xi X_{\pm\alpha_2}\\ &X_{\pm(\alpha_1+\alpha_2)} \longrightarrow \mp\xi X_{\pm(\alpha_1+\alpha_2)}\\ &X_{\pm(2\alpha_1+\alpha_2)} \longrightarrow \pm\xi X_{\pm(2\alpha_1+\alpha_2)}\end{aligned} \tag{D.4}$$

Table D.1.

	$2\alpha_1+\alpha_2$	$\alpha_1+\alpha_2$	α_2	α_1	$-\alpha_1$	$-\alpha_2$	$-\alpha_1-\alpha_2$	$-2\alpha_1-\alpha_2$
$2\alpha_1+\alpha_2$	0	0	0	0	q^{-2}	0	$-q^{-2}$	0
$\alpha_1+\alpha_2$	0	0	0	q^{-1}	q^{-3}	-1	0	$-q^{-2}$
α_2	0	0	0	q^{-2}	0	0	-1	0
α_1	0	$-q$	$-q^2$	0	0	0	q^{-3}	q^{-2}

where $\xi = -(q + q^{-1})^{1/2}(q^2 - 1 + q^{-2})$. The algebra $(C_2)_h$ is isomorphic to the algebra $(B_2)_{2h}$. The change in h is due to our choice of conventions for d_i in the definition of $U_h(\mathfrak{g})$ in appendix A.

References

[1] Aschieri P and Castellani L 1993 An introduction to noncommutative differential geometry on quantum groups *Int J. Mod. Phys.* A **8** 1667
[2] Bernard D 1990 Quantum Lie algebras and differential calculus on quantum groups *Prog. Theor. Phys. Suppl.* **102** 49
[3] Chari V and Pressley A 1994 *A Guide to Quantum Groups* (Cambridge: Cambridge University Press)
[4] Delius G W and Hüffmann A 1996 On quantum Lie algebras and quantum root systems *J. Phys. A: Math. Gen.* **29** 1703
[5] Delius G W, Hüffmann A, Gould M D and Zhang Y-Z 1996 *The quantum Lie algebras associated to $U_h(gl_n)$ and $U_h(sl_n)$ J. Phys. A: Math. Gen.* **29** 5611
[6] Delius G W and Gould M 1997 Quantum Lie algebras, their existence, uniqueness and q-antisymmetry *Commun. Math. Phys.* **185** 709
[7] Delius G W 1997 The problem of differential calculus on quantum groups *Czech. J. Phys.* **47** 1217
[8] Drinfel'd V G 1985 Hopf algebras and the quantum Yang–Baxter equation *Sov. Math. Dokl.* **32** 254
Drinfel'd V G 1986 Quantum Groups *Proc. Int. Congr. Math.* (Berkeley, CA) p 798
[9] Drinfel'd V G 1990 On almost cocommutative Hopf algebras *Leningrad Math. J.* **1**
[10] Faddeev L D and Pyatov P N 1996 *The differential calculus on quantum linear groups (AMS Transl.)* vol 175 (New York: American Mathematical Society) pp 35–47
Faddeev L D and Pyatov P N 1996 Quantization of differential calculus on linear groups *Problems in Modern Theoretical Physics* ed A P Isaev (Dubna: Joint Institute for Nuclear Research) pp 19–43
Faddeev L D and Pyatov P N 1996 Quantization of differential calculus on linear groups *Problems in Modern Theoretical Physics* ed A P Isaev (Dubna: Joint Institute for Nuclear Research) pp 96–212
[11] Gardner C 1997 *Preprint* in preparation, see http://www.mth.kcl.ac.uk/˜delius/q-lie/
[12] Humphreys J E 1970 *Introduction to Lie Algebras and Representation Theory* (Berlin: Springer)
[13] Rosso M 1990 Analogues de la forme de Killing et du theoreme d'Harish-Chandra pour les groupes quantiques *Ann. Sci. Ec. Norm. Sup.* **23** 445
[14] Samelson H 1969 *Notes on Lie Algebras* (New York: Van Nostrand)
[15] Lyubashenko V and Sudbery A *Quantum Lie Algebras of Type A_n*, q-alg/9510004
[16] Sweedler M E 1996 *Hopf algebras* (New York: Benjamin)
[17] Wolfram S 1991 *Mathematica* 2nd edn (New York: Addison-Wesley)
[18] Woronowicz S L 1989 Differential calculus on compact matrix pseudogroups (quantum groups) *Commun. Math. Phys.* **122** 125